Discrete Fractal Dimensions of the Ranges of Random Walks in $\mathbb{Z}^d$ Associate with Random Conductances

Yimin Xiao*
Michigan State University

Xinghua Zheng †
Hong Kong University of Science and Technology

February 28, 2012

Abstract

Let $X = \{X_t, t \geq 0\}$ be a continuous time random walk in an environment of i.i.d. random conductances $\{\mu_e \in [1, \infty), e \in E_d\}$, where E_d is the set of nonoriented nearest neighbor bonds on the Euclidean lattice $\mathbb{Z}^d$ and $d \geq 3$. Let $\mathrm{R} = \{x \in \mathbb{Z}^d : X_t = x \text{ for some } t \geq 0\}$ be the range of X. It is proved that, for almost every realization of the environment, $\dim_{\mathrm{H}} \mathrm{R} = \dim_{\mathrm{P}} \mathrm{R} = 2$ almost surely, where $\dim_{\mathrm{H}}$ and $\dim_{\mathrm{P}}$ denote respectively the discrete Hausdorff and packing dimension. Furthermore, given any set $A \subseteq \mathbb{Z}^d$, a criterion for A to be hit by X_t for arbitrarily large $t > 0$ is given in terms of $\dim_{\mathrm{H}} A$. Similar results for Bouchoud's trap model in $\mathbb{Z}^d$ $(d \geq 3)$ are also proven.

Running head: Fractal Dimensions of the Range of the Random Conductance Model

2010 AMS Classification numbers: 60K37, 60F17, 82C41, 31C20.

Key words: Random conductance model, Bouchoud's trap model, range, discrete Hausdorff dimension, discrete packing dimension, transience.

1 Introduction

Ordinary fractal dimensions such as Hausdorff dimension and packing dimension are useful tools not only for analyzing the (microscopic) geometric structures of various thin sets and measures in the Euclidean space $\mathbb{R}^d$, but also for many scientific applications; see Falconer [15] for a systematic account. In probability theory, they have been applied to study fine properties of the sample paths of Brownian motion, Lévy processes and random fields. We refer to Taylor [29], Xiao [31, 32] for more information. Many discrete sets, such as percolation clusters, also exhibit (macroscopic or global) fractal phenomena. In order to investigate their geometric structures, Barlow and Taylor [5, 6] have introduced the notions of discrete Hausdorff and packing dimensions and used them to study the fractal properties of strictly stable random walks. See also Khoshnevisan [19].

In this paper, we apply discrete Hausdorff and packing dimensions of Barlow and Taylor [6] to describe the range of a class of random walks in random environment, namely the random

*Research partially supported by NSF grant DMS-1006903
†Research partially supported by GRF 606010 of the HKSAR.

conductance models (RCM) on the Euclidean lattice $\mathbb{Z}^d$ considered by Barlow and Deuschel [4], among others.

More specifically, for $x, y \in \mathbb{Z}^d$, we say that $x \sim y$ if x and y are neighboring sites, (i.e., $|x-y|=1$, where $|\cdot|$ is the Euclidean distance) and $x \not\sim y$ otherwise. Let E_d be the set of nonoriented nearest neighbor bonds, i.e., $E_d = \{e=(x,y):\ x \sim y\}$, and let $\{\mu_e, e \in E_d\}$ be a sequence of non-negative i.i.d. random variables with values in $[1,\infty)$, defined on a probability space $(\Omega, \mathbb{P})$. We may take $\Omega = [1,\infty)^{E_d}$, the set of configurations of conductances, and let $\mathbb{P}$ be the product probability measure on Ω under which the coordinates $\mu_e, e \in E_d$, are i.i.d. random variables.

We write $\mu_{xy} = \mu_{(x,y)} = \mu_{yx}$, let $\mu_{xy} = 0$ if $x \nsim y$ and set $\mu_x = \sum_y \mu_{xy}$. There are two natural continuous time random walks on $\mathbb{Z}^d$ associated with $\{\mu_e, e \in E_d\}$. Both jump from x to $y \sim x$ with probability $P(x,y) = \mu_{xy}/\mu_x$. The first (the variable speed random walk or VSRW) waits at x for an exponential time with mean $1/\mu_x$ while the second (the constant speed random walk or CSRW) waits at x for an exponential time with mean 1. Their generators $\mathcal{L}_V$ and $\mathcal{L}_C$ are given by

$$\begin{aligned} \mathcal{L}_V(\omega) f(x) &= \sum_y \mu_{xy}(\omega)\big(f(x)-f(y)\big), \quad \text{and} \\ \mathcal{L}_C(\omega) f(x) &= \mu_x(\omega)^{-1} \sum_y \mu_{xy}(\omega)\big(f(x)-f(y)\big), \end{aligned} \tag{1.1}$$

respectively. VSRW is reversible with stationary measure ν defined by $\nu(\{x\}) = 1$, $x \in \mathbb{Z}^d$; and CSRW is reversible with $\mu_x, x \in \mathbb{Z}^d$ as its stationary measure. Since the generators $\mathcal{L}_V$ and $\mathcal{L}_C$ only differ by a multiple, VSRW and CSRW are time change of each other; see Barlow and Deuschel [4, pp.39-40] for precise information.

The random conductance model has been studied by several authors under various restrictions on the law of μ_e. There are three typical cases: $c^{-1} \le \mu_e \le c$ for some $c \ge 1$ (strong ellipticity), $0 \le \mu_e \le 1$, and $1 \le \mu_e < \infty$. An important example of the random conductance models is a continuous time simple random walk on a supercritical percolation cluster $\mathcal{C}_\infty$ in $\mathbb{Z}^d$. In this case $\{\mu_e, e \in E_d\}$ are i.i.d. Bernoulli random variables with mean $p > p_c(d)$, the critical probability for bond percolation in $\mathbb{Z}^d$. See Barlow [2], Berger et al. [9], Biskup and Prescott [10], Mathieu [23] and the references therein for further information.

Under the assumptions that $d \ge 2$ and $\mathbb{P}(\mu_e \ge 1) = 1$, Barlow and Deuschel [4] prove that the VSRW satisfies a quenched functional central limit theorem and the limiting process is $\sigma_V B$, where $\sigma_V > 0$ is a nonrandom constant and B is a Brownian motion on $\mathbb{R}^d$. As shown by Barlow and Černý [3], Barlow and Zheng [7] and Černý [11], the scaling limit of CSRW in $\mathbb{Z}^d$ with $d \ge 2$ in the heavy-tailed environment can either be a Brownian motion or a fractional-kinetics process (which is a Brownian motion time-changed by the inverse of a stable subordinator). Random conductance models under the general conditions $\mathbb{P}\big(0 \le \mu_e < \infty\big) = 1$ and $\mathbb{P}\big(\mu_e > 0\big) > p_c(d)$ have been recently investigated by Andres et al. [1]. In this paper we focus on the case $\mathbb{P}(\mu_e \ge 1) = 1$.

This paper is concerned with fractal properties of the ranges of random conductance models. Since the time change which relates CSRW and VSRW is strictly increasing and continuous, VSRW and CSRW have the same range. Hence, in the following, we consider VSRW in $\mathbb{Z}^d$ and denote it by X. Also, for any environment $\{\mu_e(\omega), e \in E_d\}$ and any $x \in \mathbb{Z}^d$, we write P_x^ω for the (quenched) law of X started at x.

Let

$$\mathrm{R} = \{x \in \mathbb{Z}^d : X_t = x \text{ for some } t \geq 0\}$$

be the range of VSRW X in $\mathbb{Z}^d$. It follows from Theorem 1.2 of Barlow and Deuschel [4] that when $d \geq 2$ X is transient if and only if $d \geq 3$. Hence for $d = 2$, X is recurrent and $\mathrm{R} = \mathbb{Z}^2$ P_0^ω-a.s. The case of $d = 1$ is similar because by, for example, Lemma 1.5 of Solomon [27], the range is almost surely the whole line. We shall henceforth assume that $d \geq 3$.

The following are our main theorems, which describe the fractal structures of R and characterize the transient sets for X by using the discrete Hausdorff and packing dimensions defined by Barlow and Taylor [5, 6].

Theorem 1.1 *Assume that $d \geq 3$ and $\mathbb{P}(\mu_e \geq 1) = 1$. Then for $\mathbb{P}$-almost every $\omega \in \Omega$,*

$$\dim_{\mathrm{H}} \mathrm{R} = \dim_{\mathrm{P}} \mathrm{R} = 2, \quad \mathrm{P}_0^\omega\text{-}a.s.,$$

where $\dim_{\mathrm{H}}$ and $\dim_{\mathrm{P}}$ denote respectively the discrete Hausdorff and packing dimension.

Theorem 1.2 *Assume that $d \geq 3$ and $\mathbb{P}(\mu_e \geq 1) = 1$. Let $A \subset \mathbb{Z}^d$ be any (infinite) set. Then for $\mathbb{P}$-almost every $\omega \in \Omega$, the following statements hold.*

(i) If $\dim_{\mathrm{H}} A < d - 2$, then

$$\mathrm{P}_0^\omega\big(X_t \in A \ \ \textit{for arbitrarily large } t > 0\big) = 0.$$

(ii) If $\dim_{\mathrm{H}} A > d - 2$, then

$$\mathrm{P}_0^\omega\big(X_t \in A \ \ \textit{for arbitrarily large } t > 0\big) = 1.$$

The above theorems show that, if $\mu_e \geq 1$, then for almost every realization of the environment, VSRW and CSRW have long term fractal and asymptotic behavior similar to the simple random walk on $\mathbb{Z}^d$ and Brownian motion in $\mathbb{R}^d$.

Remark 1.3

(i) When $c^{-1} \leq \mu_e \leq c$ for some constant $c \geq 1$, Barlow and Deuschel [4, Remark 6.3 and Theorem 6.1] prove that, if $\{\mu_e, e \in E_d\}$ is stationary, symmetric and ergodic, then Lemmas 2.1 and 3.9 below still hold. (See also Delmott [14] for i.i.d. environment.) As the proofs of Theorems 1.1 and 1.2 only make use of Lemmas 2.1 and 3.9, it follows that in this case the independence assumption on $\{\mu_e\}$ in Theorems 1.1 and 1.2 can be weakened and the same conclusions remain valid.

(ii) When $\mathbb{P}\big(0 \leq \mu_e \leq 1\big) = 1$, only partial estimates on the heat kernel of X on the diagonal are available, see [9, 10, 23]. Berger et al. [9] show that Gaussian heat kernel bounds do not hold in general. This is caused by traps due to edges in E_d with small positive conductances.

Under the extra condition that $\mathbb{P}\big(\mu_e > 0\big) > p_c(d)$, (otherwise the range R of X is a finite set,) Andres et al. [1] prove that the Green's function of X satisfies bounds in (d) of Lemma 2.1 below, but their Remark 7.6 shows that (e) of Lemma 2.1 does not hold in general. Since our proofs of Theorems 1.1 and 1.2 rely heavily on Lemma 2.1, it is not known whether similar results still hold. We will consider these and related problems separately.

□

The proofs of Theorems 1.1 and 1.2 are similar to those of Theorems 7.8 and 8.3 of Barlow and Taylor [6], where transient, strictly α-stable random walks on $\mathbb{Z}^d$ are treated. However, there are significant differences between VSRW and strictly stable random walks. One major difference is that VSRW is not a random walk in the classical sense since it does not have i.i.d. increments. We make use of general Markov techniques to derive hitting probability estimates and maximal inequality for VSRW, and also to overcome the difficulties caused by the dependence between the increments, see, e.g., Lemma 2.6. These results and the proofs of Theorems 1.1 and 1.2 are given in Section 3. Since our arguments are based on general Markovian techniques, they will be useful for studying other properties of VSRW, as well as more general Markov chains.

We also consider another kind of random walk in random environment, namely Bouchaud's trap model (BTM). This model was first introduced in the physics literature to explain some strange dynamical properties of complex disordered systems, in particular aging. We refer to Barlow and Černý [3] for a brief historical account on BTM and to Ben Arous and Černý [8], Barlow and Černý [3] and Černý [11] for results on scaling limits.

To recall the definition of BTM, let $\{\kappa_x, x \in \mathbb{Z}^d\}$ be i.i.d. positive random variables on a probability space $(\tilde{\Omega}, \tilde{\mathbb{P}})$. For a given constant $a \in [0, 1]$, define random conductances $\tilde{\mu}_e$ ($e \in E_d$) on $\mathbb{Z}^d$ by

$$\tilde{\mu}_{xy} = \kappa_x^a \kappa_y^a, \qquad \text{if } \ x \sim y.$$

Then BTM is the continuous time Markov chain on $\mathbb{Z}^d$ whose transition rates w_{xy} are given by

$$w_{xy} = \frac{\tilde{\mu}_{xy}}{\kappa_x} = \kappa_x^{a-1} \kappa_y^a, \qquad \text{if } \ x \sim y.$$

If $a = 0$, then $\tilde{\mu}_e = 1$ for all $e \in E_d$ and the BTM is a time change of the simple random walk on $\mathbb{Z}^d$. If $a \neq 0$, then, following Barlow and Černý [3], it is referred to as the non-symmetric BTM.

Just in the same way as for the RCM, we can define the VSRW, denoted by $\tilde{X}$, associated with the conductances $\{\tilde{\mu}_e\}$. The BTM and $\tilde{X}$ are again related to each other by a time change, see equation (2.3) in [3], and so in particular have the same range. Similarly as before, for any environment $\{\tilde{\mu}_e(\omega), e \in E_d\}$ and any $x \in \mathbb{Z}^d$, we write $\tilde{\mathrm{P}}_x^\omega$ for the (quenched) law of $\tilde{X}$ started at x.

Even though the conductances $\{\tilde{\mu}_e, e \in E_d\}$ are not independent any more, they form a stationary symmetric ergodic process. By applying the results in Barlow and Deuschel [4] and Barlow and Černý [3] to the VSRW $\tilde{X}$, we can use the method in Section 3 to prove the following theorem.

Theorem 1.4 *Assume that $d \geq 3$ and $\tilde{\mathbb{P}}\big(\kappa_x \geq 1\big) = 1$. Let $\tilde{\mathrm{R}}$ be the range of the Bouchaud's trap model. Then for $\tilde{\mathbb{P}}$-almost every $\omega \in \tilde{\Omega}$,*

$$\dim_{\mathrm{H}} \tilde{\mathrm{R}} = \dim_{\mathrm{P}} \tilde{\mathrm{R}} = 2, \quad \tilde{\mathrm{P}}_0^\omega\text{-}a.s.$$

Moreover, the conclusions of Theorem 1.2 hold for $\tilde{X}$.

The proof of Theorem 1.4 is given in Section 4. Similarly to Remark 1.3 (ii), it would be interesting to determine whether the assumption that κ_x is bounded from below can be removed.

As is mentioned by an anonymous referee, in light of the above results it would be interesting to investigate the discrete fractal dimensions of percolation clusters and the images $X(E)$, where $E \subset \mathbb{R}_+$ and X is VSRW or BTM, or an ordinary α-stable random walk as in Barlow and Taylor [6]. We thank him/her for his/her thoughtful suggestions, and we will study these questions in subsequent work.

Throughout this paper, for any $x, y \in \mathbb{Z}^d$, $|x-y|$ stands for the Euclidean distance, and the ℓ_∞ distance is denoted by $||x-y||_\infty = \max_{i=1}^d |x_i - y_i|$. We will use c, c', c'' etc to denote unspecified positive and finite (nonrandom) constants, which may depend on the distribution of the environment and may be different in each occurrence. More specific constants are numbered as $c_1, c_2, \cdots$.

2 Preliminaries

In this section, we recall some known facts about the VSRW and discrete Hausdorff and packing dimensions, and prove a strong law of large numbers (SLLN) for dependent events, which will be used in this paper.

2.1 Some basic properties of VSRW

Let $X = \{X_t, t \geq 0\}$ be a VSRW with values in $\mathbb{Z}^d$ with $d \geq 3$ and let $p_t^\omega(x,y) = \mathrm{P}_x^\omega(X(t) = y)$ be its transition density or the heat kernel of $\mathcal{L}_V$.

The following estimates for the transition density $p_t^\omega(x,y)$ and the Green's function $g_t^\omega(x,y)$ will be used in the sequel. Recall that when $d \geq 3$, X is transient and $g_t^\omega(x,y)$ is defined by

$$g_t^\omega(x,y) = \int_0^\infty p_t^\omega(x,y)\, dt.$$

Lemma 2.1 *Let $d \geq 3$, $\mathbb{P}(\mu_e \geq 1) = 1$, and $\eta \in (0,1)$. There exist random variables $\{U_x, x \in \mathbb{Z}^d\}$ and positive (nonrandom) constants c_i (depending on d and the distribution of μ_e) such that*

$$\mathbb{P}\big(U_x \geq n\big) \leq c_1 \exp\big(-c_2 n^\eta\big).$$

(a) [4, Theorem 1.2(a)] *For all $x, y \in \mathbb{Z}^d$ and $t > 0$,*

$$p_t^\omega(x,y) \leq 1 \wedge \big(c_3 t^{-d/2}\big).$$

(b) [4, Theorem 1.2(b)] *If $|x-y| \vee \sqrt{t} \geq U_x$, then*

$$p_t^\omega(x,y) \leq \begin{cases} c_4 t^{-d/2} \exp(-c_5 |x-y|^2/t), & \text{when } t \geq |x-y|, \\ c_4 \exp\big(-c_5|x-y|(1 \vee \log(|x-y|/t))\big), & \text{when } t \leq |x-y|. \end{cases}$$

(c) [4, Theorem 1.2(c)] *If $t \geq U_x^2 \vee |x-y|^{1+\eta}$, then*

$$p_t^\omega(x,y) \geq c_6 t^{-d/2} \exp(-c_7 |x-y|^2/t).$$

(d) [4, Theorem 1.3] *If* $|x-y| \geq U_x \wedge U_y$, *then*

$$\frac{c_8}{|x-y|^{d-2}} \leq g^\omega(x,y) \leq \frac{c_9}{|x-y|^{d-2}}.$$

(e) [3, Lemma 3.4] *For all* $x, y \in \mathbb{Z}^d$,

$$g^\omega(x,y) \leq c_{10}.$$

(f) [3, Lemma 3.3] *There exists* $c_{11} > 0$ *such that for each* $K > 0$, *the inequality*

$$\max_{|x| \leq Kn} U_x \leq c_{11} (\log n)^{1/\eta} \tag{2.1}$$

holds with $\mathbb{P}$*-probability no less than* $1 - c_{12} K^d n^{-2}$. *In particular,* $\mathbb{P}$*-a.s. there exists* $n_0 = n_0(\omega)$ *such that* (2.1) *holds for all* $n \geq n_0$.

In the rest of this paper, we take $\eta = 1/3$. Hence $\mathbb{P}$-a.s. there exists $n_0 = n_0(\omega)$ such that

$$\max_{||x||_\infty \leq 2^n} U_x \leq c_{11}\, n^3 \qquad \text{for all } \ n \geq n_0. \tag{2.2}$$

We will sometimes work with the *discrete time* VSRW $\widehat{X} = \{\widehat{X}_n : n = 0, 1, \ldots\}$ defined by $\widehat{X}_n = X_n$ for $n = 0, 1, \ldots$. Its transition probabilities are nothing but $p^\omega_n(x,y)$, so in particular, satisfy (a), (b), (c) of the previous lemma. Define its Green's function as

$$\widehat{g}^\omega(x,y) = \sum_{n=0}^{\infty} p^\omega_n(x,y). \tag{2.3}$$

Then using (a), (b) and (c) of Lemma 2.1 and similar computations as in §4.3 of Lawler and Limic [21] one can derive

Lemma 2.2 *When* $d \geq 3$, *the inequalities in (d) and (e) of Lemma 2.1 also hold for* $\widehat{g}^\omega(x,y)$.

We also recall the following connection between the hitting probabilities of a time homogeneous transient Markov chain $\{X_t,\, t \geq 0, \mathrm{P}_x, x \in E\}$ on a discrete state space E and the capacity with respect to its Green's function $g(x,y)$.

It is known that for any finite set $A \subseteq E$, there is a positive function $b(\cdot)$ supported by A such that

$$\mathrm{P}_x\big(X_t \in A \ \text{ for some } \ t \geq 0\big) = \sum_{y \in A} g(x,y) b(y). \tag{2.4}$$

This follows from Chung [12, 13]. For an explicit expression of the function $b(\cdot)$, see also Syski [28, p.435] or the proof of Lemma 2.3 in the Appendix.

The natural capacity of A with respect to g is defined by

$$\mathrm{Cap}_g(A) = \sum_{y \in A} b(y). \tag{2.5}$$

For any measure σ on A, write $(g\sigma)(x) = \sum_{y \in A} g(x,y)\sigma(y)$ for the potential due to the charge σ. Then we have

Lemma 2.3 *If the time homogeneous transient Markov chain $\{X_t,\, t \geq 0\}$ has a discrete state space E, is right continuous, and satisfies the following conditions:*

(i) $p_t(x, y) \leq f(t)$ for all $x, y \in E$, where the function f may depend on (x, y) and is decreasing and integrable on $[0, \infty)$;

(ii) for any $x \in E$, the rate q_x of leaving x is finite.

Then for any finite set $A \subseteq E$,

$$\mathrm{Cap}_g(A) = \max\Big\{\sigma(A) : \sigma \text{ is a measure on } A \text{ such that } \max_{x\in A}(g\sigma)(x) \leq 1\Big\}. \tag{2.6}$$

In particular, (2.6) *holds for the VSRW.*

It follows from (2.6) that the capacity of a singleton $\{x\}$ is $g(x, x)^{-1}$.

As Lemma 2.3 is (more or less) well known (but we didn't succeed in finding a version similar to what is stated above that fits our needs), we shall only briefly sketch its proof in the Appendix.

2.2 Discrete fractal dimensions

We recall briefly the definitions and basic properties of fractal dimensions for subsets of $\mathbb{Z}^d$ from Barlow and Taylor [6].

For $x \in \mathbb{Z}^d$ and $n \geq 1$, define cubes

$$\begin{aligned} C(x, n) &= \{y \in \mathbb{Z}^d : x_i \leq y_i < x_i + n\}, \quad \text{and} \\ V(x, n) &= \big\{y \in \mathbb{Z}^d : x_i - \frac{1}{2}n \leq y_i < x_i + \frac{1}{2}n\big\}. \end{aligned} \tag{2.7}$$

Clearly $C(x, 1) = V(x, 1) = \{x\}$ and $\#\big(C(x, n)\big) = \#\big(V(x, n)\big) = n^d$. Here and in the sequel, $\#(A)$ denotes the cardinality of A.

Denote by $\mathcal{C}, \mathcal{C}_d$ and $\mathcal{C}_s$ the classes of cubes, dyadic cubes and semi-dyadic cubes in $\mathbb{Z}^d$. Namely,

$$\begin{aligned} \mathcal{C} &= \big\{C(x, n) : x \in \mathbb{Z}^d, n \geq 1\big\}, \\ \mathcal{C}_d &= \big\{C(x, 2^n) : x \in 2^n\mathbb{Z}^d, n \geq 1\big\}, \quad \text{and} \\ \mathcal{C}_s &= \big\{V(x, 2^n) : x \in 2^{n-1}\mathbb{Z}^d, n \geq 1\big\}. \end{aligned} \tag{2.8}$$

The side of $A \subseteq \mathbb{Z}^d$, denoted by $s(A)$, is defined by

$$s(A) = \inf\big\{r > 0 :\ A \subseteq C(x, r) \text{ for some } x \in \mathbb{Z}^d\big\}.$$

Let $\mathcal{C}_d^k$ and $\mathcal{C}_s^k$ denote the classes of dyadic and semi-dyadic cubes of side 2^k. Note that each $x \in \mathbb{Z}^d$ belongs to a unique cube in $\mathcal{C}_d^k$, which is denoted by $Q_k(x)$. Each $x \in \mathbb{Z}^d$ belongs to 2^d cubes in $\mathcal{C}_s^k$ and we write $\widetilde{V}(x, 2^k)$ for the semi-dyadic cube $V \in \mathcal{C}_s^k$ with center closest to x.

Let $V_n = V(0, 2^n)$ for all $n \geq 0$, $S_1 = V_1$ and $S_n = V_n \backslash V_{n-1}$ for $n \geq 2$. Thus $\{S_n, n \geq 2\}$ is a sequence of disjoint cubical shells centered on the point $(-\frac{1}{2}, \ldots, -\frac{1}{2})$.

Let $\mathcal{H}$ be the collection of functions $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that h is continuous, monotone increasing, $h(0) = 0$, and satisfies $h(2r) \leq c_h\, h(r)$ for all $r \in [0, 1/2]$, where c_h is a constant. Functions in $\mathcal{H}$ are called measure functions.

For any $A \subseteq \mathbb{Z}^d$ and $h \in \mathcal{H}$, set

$$\nu_h(A, S_n) = \min\Big\{ \sum_{i=1}^{k} h\Big(\frac{s(B_i)}{2^n}\Big) : B_i \in \mathcal{C},\, A \cap S_n \subset \bigcup_{i=1}^{k} B_i \Big\}. \tag{2.9}$$

The discrete Hausdorff measure of A with respect to the measure function h is defined by

$$m_h(A) = \sum_{n=1}^{\infty} \nu_h(A, S_n). \tag{2.10}$$

If $h(r) = r^\alpha$ ($\alpha > 0$), we write ν_α and m_α for ν_h and m_h, respectively. The discrete Hausdorff dimension of A is defined by

$$\dim_{\rm H} A = \inf\big\{\alpha > 0 : m_\alpha(A) < \infty\big\}. \tag{2.11}$$

It is often more convenient to replace $\mathcal{C}$ in (2.9) by the smaller class $\mathcal{C}_d$, the corresponding values to (2.9) and (2.10) will be written as $\widetilde{\nu}_h(A, S_n)$ and $\widetilde{m}_h(A)$, respectively. Barlow and Taylor [6, p.128] proved that $\nu_h(A, S_n) \le \widetilde{\nu}_h(A, S_n) \le 2^d \nu_h(A, S_n)$. Hence $m_h(A)$ and $\widetilde{m}_h(A)$ are comparable, and replacing m_α in (2.11) by $\widetilde{m}_\alpha$ defines the same $\dim_{\rm H} A$.

Discrete packing measure and packing dimension of A are defined in a dual way. For any $h \in \mathcal{H}$ and $\varepsilon > 0$, define

$$\tau_h(A, S_n, \varepsilon) = \max\Big\{ \sum_{i=1}^{k} h\Big(\frac{r_i}{2^n}\Big) : x_i \in A \cap S_n, V(x_i, r_i) \text{ disjoint, } 1 \le r_i \le 2^{(1-\varepsilon)n} \Big\} \tag{2.12}$$

and

$$p_h(A, \varepsilon) = \sum_{n=1}^{\infty} \tau_h(A, S_n, \varepsilon). \tag{2.13}$$

A set $A \subseteq \mathbb{Z}^d$ is said to be *h-packing finite* if $p_h(A, \varepsilon) < \infty$ for all $\varepsilon \in (0, 1)$. Again, if $h(r) = r^\alpha$, we write τ_h and p_h as τ_α and p_α.

The discrete packing dimension of A is defined by

$$\dim_{\rm P} A = \inf\big\{\alpha > 0 : A \text{ is } r^\alpha\text{-packing finite}\big\}. \tag{2.14}$$

One can use semi-dyadic cubes in (2.12) and define

$$\widetilde{\tau}_h(A, S_n, \varepsilon) = \max\Big\{ \sum_{i=1}^{k} h\Big(\frac{2^{k_i}}{2^n}\Big) : x_i \in A \cap S_n, \widetilde{V}(x_i, 2^{k_i}) \text{ disjoint, } 2^{k_i} \le 2^{(1-\varepsilon)n} \Big\} \tag{2.15}$$

and the corresponding $\widetilde{p}_h(A, \varepsilon)$. [6, p.130] proved that there exists a constant $c > 1$ (depending on h) such that

$$c^{-1}\tau_h(A, S_n, \varepsilon) \le \widetilde{\tau}_h(A, S_n, \varepsilon) \le c\, \tau_h(A, S_n, \varepsilon)$$

for all A, n and $\varepsilon \in (0, 1)$. Thus, A is h-packing finite if and only if $\widetilde{p}_h(A, \varepsilon) < \infty$ for all $\varepsilon \in (0, 1)$.

From (2.11) and (2.14) it is clear that $\dim_{\rm H} A$ and $\dim_{\rm P} A$ do not depend on the part of A which lies inside any ball of finite radius. They are determined by the geometric structure of A

at infinity. Similarly to the ordinary Hausdorff and packing dimensions in $\mathbb{R}^d$, the discrete Hausdorff and packing dimensions on $\mathbb{Z}^d$ satisfy the following relationship: For all $A \subseteq \mathbb{Z}^d$,

$$0 \le \dim_{\mathrm{H}} A \le \dim_{\mathrm{P}} A \le d, \tag{2.16}$$

and the inequalities may be strict. See Barlow and Taylor [6, pp.130, 132 and 136].

It is often not difficult to find optimal covering or packing for $A \cap S_n$, which leads to good upper bound for $\nu_h(A \cap S_n)$ and lower bound for $\tau_h(A, S_n, \varepsilon)$. However, a direct approach for obtaining the lower bound for $\nu_h(A \cap S_n)$ (or upper bound for $\tau_h(A, S_n, \varepsilon)$) is usually tricky. The following lemmas are useful. The first is an analogue of the density lemma and is a consequence of Theorem 4.1 of Barlow and Taylor [6]. The second is an analogue of Frostman's lemma and follows from Theorem 4.2 of Barlow and Taylor [6].

Lemma 2.4 *Let $h \in \mathcal{H}$ and μ be a measure on $A \subseteq S_n$. If*

$$\mu(A \cap V(x, 2^k)) \le a_1\, h\big(2^{k-n}\big) \qquad \text{for all } x \in \mathbb{Z}^d,\ 0 \le k \le n.$$

Then $\nu_h(A,\, S_n) \ge 2^{-d} a_1^{-1} \mu(A)$.

Lemma 2.5 *Let $h \in \mathcal{H}$ and $A \subseteq S_n$. Then there is a measure μ on A that satisfies*

$$\mu(A) \ge \nu_h(A, S_n) \qquad \text{and} \qquad \mu(V(x, 2^k)) \le 2^d\, h\Big(\frac{2^k}{2^n}\Big) \ \text{ for all } 0 \le k \le n,\, x \in A.$$

2.3 A SLLN for dependent events

The increments of VSRW are not independent. For this reason, we here establish a SLLN for dependent events which will be used in our proof of Theorem 1.1 below.

Lemma 2.6 *Suppose that $\{A_i\}, \{B_i\}$ are two sequences of events adapted to a (common) filtration $\{\mathcal{F}_i\}$ and are such that for some positive constants p, a and δ*

$$\mathrm{P}(A_{i+1}|\mathcal{F}_i) \ge p \quad \text{on event } B_i, \text{ and } \ \mathrm{P}(B_i^c) \le a e^{-\delta i} \ \text{ for all } i. \tag{2.17}$$

Write $X_i = \mathbb{1}_{A_i}$, and $S_n = \sum_{i=1}^n X_i$. Then there exists $\varepsilon > 0$ such that

$$\liminf_{n\to\infty} \frac{S_n}{n} \ge \varepsilon \qquad \text{almost surely.}$$

Proof We first estimate the moment generating function of S_n. For any $t > 0$,

$$\begin{aligned} \mathrm{E}(e^{-tS_n}) &= \mathrm{E}\big(\mathrm{E}(e^{-tX_n}|\mathcal{F}_{n-1}) \cdot e^{-tS_{n-1}}\big) \\ &\le \mathrm{E}\big(\mathrm{E}(e^{-tX_n}|\mathcal{F}_{n-1}) \cdot \mathbb{1}_{B_{n-1}} \cdot e^{-tS_{n-1}}\big) + \mathrm{P}(B_{n-1}^c). \end{aligned} \tag{2.18}$$

By using the first inequality in (2.17) and the elementary inequality $1 - x \le e^{-x}$ ($x \ge 0$), we derive that

$$\mathrm{E}\big(e^{-tX_n}|\mathcal{F}_{n-1}\big) \le q(t) := e^{-p(1-e^{-t})} < 1 \quad \text{ on event } B_{n-1}. \tag{2.19}$$

Now choose $k > 0$ large enough and $b > 0$ small enough such that

$$b \le \delta, \quad k e^{-b} \ge 1, \quad \text{and} \quad q(t) e^b + \frac{a e^\delta}{k} \le 1.$$

We go on to show that

$$\mathrm{E}(e^{-tS_n}) \le k e^{-bn}, \quad \text{for all } n. \tag{2.20}$$

In fact, by the choices of k and b, (2.20) holds automatically for $n = 1$. Now suppose that it holds for $n-1$, then by (2.18) and (2.19) and using induction one gets that

$$\mathrm{E}(e^{-tS_n}) \le q(t)\mathrm{E}(e^{-tS_{n-1}}) + \mathrm{P}(B_{n-1}^c) \le q(t)ke^{-b(n-1)} + ae^{\delta} \cdot e^{-\delta n}.$$

The last term is bounded by ke^{-bn}, by the choices of k and b.

Once we have the bound (2.20) for $\mathrm{E}(e^{-tS_n})$, the conclusion then follows easily by using the Chebyshev's inequality and the Borel-Cantelli lemma. □

3 Proofs of Theorems 1.1 and 1.2

As in Barlow and Taylor [6], the proof of Theorem 1.1 is divided into proving the upper bound $\dim_{\mathrm{P}}\mathrm{R} \le 2$ P_0^ω-a.s. and the lower bound $\dim_{\mathrm{H}}\mathrm{R} \ge 2$ P_0^ω-a.s., separately. The upper bound is proved by using a first moment argument and the lower bound is proved by using a "mass distribution" method. However, since there are significant differences between VSRW and the strictly stable random walks in Barlow and Taylor [6], some preparations are needed.

In the following we establish quenched results on hitting probability, sojourn time, maximal inequality, and a zero-one law for VSRW. These results may also be useful for studying other properties of VSRW.

3.1 Hitting probability estimates

We start with the following lemma. Its proof is a slight modification of that of Proposition 2.1 in Xiao [30] which is an extension of Theorem 1 in Khoshnevisan [20] for Lévy processes.

Lemma 3.1 *Let $\{X_t,\, t \ge 0, \mathrm{P}_x, x \in \mathbb{Z}^d\}$ be a time homogeneous (continuous time) Markov chain. Then for any $x, y \in \mathbb{Z}^d$, $b > a \ge 0$ and $r > 0$,*

$$\begin{aligned} \frac{1}{2}\frac{\int_a^b \mathrm{P}_x\big(X_t \in C(y,r)\big)\,dt}{\sup_{z\in C(y,r)}\int_0^b \mathrm{P}_z\big(X_t \in C(y,r)\big)\,dt} &\le \mathrm{P}_x\Big(X_t \in C(y,r) \ \ \text{for some } \ a \le t \le b\Big) \\ &\le \frac{\int_a^{2b-a} \mathrm{P}_x\big(X_t \in C(y,r)\big)\,dt}{\inf_{z\in C(y,r)}\int_0^{b-a} \mathrm{P}_z\big(X_t \in C(y,r)\big)\,dt}. \end{aligned} \tag{3.1}$$

Observe that, if $\int_0^\infty \mathrm{P}_x\big(X_s \in C(y,r)\big)ds < \infty$, then we can take $a = 0$ and $b = \infty$ in Lemma 3.1.

Now we apply Lemma 3.1 to derive the following hitting probability estimates for the VSRW X.

Lemma 3.2 *Assume that $d \ge 3$ and $\mathbb{P}\big(\mu_e \ge 1\big) = 1$. Then $\mathbb{P}$-a.s. for all n large enough, $r \in [n^{3d/2},\ 2^{n-1}]$, $y \in V(0, 2^n)$, and all $x \in \mathbb{Z}^d$ such that $||x-y||_\infty \ge 2r$ we have*

$$\mathrm{P}_x^\omega\Big(X_t \in C(y,r) \ \ \text{for some } \ t > 0\Big) \asymp \left(\frac{r}{|x-y|}\right)^{d-2}. \tag{3.2}$$

Here and in the sequel, $f \asymp g$ means that the ratio f/g is bounded from below and above by positive and finite constants which are independent of the variables involved (x, y and r in this case).

Proof We will apply Lemma 3.1 with $a = 0$ and $b = \infty$. We first consider the denominators in (3.1) and show that there exists a constant $c > 1$ such that

$$c^{-1} r^2 \leq \int_0^\infty \mathrm{P}_z^\omega\big(X_s \in C(y,r)\big)\, ds \leq c\, r^2 \tag{3.3}$$

for all $z \in C(y,r)$ and for all y and r that we consider.

By Fubini's theorem, we write the integral in (3.3) in terms of the Green's function of X:

$$\int_0^\infty \mathrm{P}_z^\omega\big(X_s \in C(y,r)\big)\, ds = \sum_{w \in C(y,r)} g^\omega(z,w). \tag{3.4}$$

By (2.2), for all n sufficiently large and for all $r \leq 2^{n-1}$,

$$\max_{y \in V(0,2^n),\ z \in C(y,r)} U_z \leq c_{11}\, n^3 \leq n^{3d/2}/4. \tag{3.5}$$

Hence we can apply (d) and (e) of Lemma 2.1 to (3.4) and obtain that for every $z \in C(y,r)$ with $r \geq n^{3d/2}$,

$$\begin{aligned}\int_0^\infty \mathrm{P}_z^\omega\big(X_s \in C(y,r)\big)\, ds &\leq \sum_{\{|w-z| \leq c_{11} n^3\}} c_{10} + \sum_{\{c_{11} n^3 \leq |w-z| \leq \sqrt{d}\, r\}} \frac{c_9}{|z-w|^{d-2}} \\ &\leq c\left(n^{3d} + r^2\right) \leq c\, r^2.\end{aligned} \tag{3.6}$$

This proves the upper bound in (3.3). On the other hand, since we only consider large r's, for any $z \in C(y,r)$,

$$\#\Big\{w \in C(y,r) : \frac{r}{4} \leq |w-z| \leq \frac{3r}{4}\Big\} \asymp r^d.$$

Denote the above set by Γ. Then by using (3.4), (3.5) and (d) of Lemma 2.1 again, we have

$$\int_0^\infty \mathrm{P}_z^\omega\big(X_s \in C(y,r)\big)\, ds \geq \sum_{w \in \Gamma} \frac{c_8}{|z-w|^{d-2}} \geq c^{-1}\, r^2, \tag{3.7}$$

which proves the lower bound in (3.3).

To estimate the numerators in (3.1), noting that for all $w \in C(y,r)$, since $||x-y||_\infty \geq 2r$ and hence $|x-w| \geq ||x-w||_\infty \geq r \geq \max_{w \in C(y,r)} U_w$, we can use again (d) of Lemma 2.1 to get

$$\begin{aligned}\int_0^\infty \mathrm{P}_x^\omega\big(X_t \in C(y,r)\big)\, dt &= \sum_{w \in C(y,r)} g^\omega(x,w) \\ &\asymp \sum_{w \in C(y,r)} \frac{1}{|x-w|^{d-2}} \asymp \frac{r^d}{|x-y|^{d-2}},\end{aligned} \tag{3.8}$$

where in the last step we used again that $|x-y| \geq ||x-y||_\infty \geq 2r$. Hence (3.2) follows from Lemma 3.1, (3.3) and (3.8). □

Similarly, for the discrete time VSRW $\{\widehat{X}_n,\, n \geq 0\}$, we have the following estimate regarding the hitting probabilities.

Lemma 3.3 *Assume that $d \geq 3$ and $\mathbb{P}(\mu_e \geq 1) = 1$. Then $\mathbb{P}$-a.s. for all n large enough, for all $x \in V(0, 2^{n-2})$, and for all $y \in S_n (= V(0, 2^n) \backslash V(0, 2^{n-1}))$,*

$$\mathrm{P}_x^\omega\Big(\widehat{X}_i = y \ \ \textit{for some} \ \ i \geq 0\Big) \geq \frac{c_8}{c_{10}|x-y|^{d-2}}. \tag{3.9}$$

Proof By the strong Markov property,

$$\widehat{g}^\omega(x,y) = \mathrm{P}_x^\omega\Big(\widehat{X}_i = y \ \ \text{for some} \ \ i \geq 0\Big) \cdot \widehat{g}^\omega(y,y).$$

The conclusion then follows from Lemma 2.2 and (2.2). □

Our next lemma, which is similar to Proposition 8.1 in Barlow and Taylor [6], establishes a connection between the capacity Cap_{g^ω} associated with VSRW X and Hausdorff measures.

Lemma 3.4 *Assume that $d \geq 3$ and $\mathbb{P}(\mu_e \geq 1) = 1$. Then there exists a constant $c_{13} \geq 1$ such that for $\mathbb{P}$-almost every $\omega \in \Omega$, all n large enough and sets $A \subseteq S_n$,*

$$c_{13}^{-1}\, 2^{n(d-2)}\, \nu_{h_2}(A, S_n) \leq \mathrm{Cap}_{g^\omega}(A) \leq c_{13}\, 2^{n(d-2)}\, \nu_{h_1}(A, S_n). \tag{3.10}$$

In the above,

$$h_1(r) = \begin{cases} r^{d-2}\Big(\log\Big(\dfrac{1}{r}\Big)\Big)^{3d(d-2)/2}, & \textit{if } r \leq r_0; \\[2ex] r^{d-2}\Big(\log\Big(\dfrac{1}{r_0}\Big)\Big)^{3d(d-2)/2}, & \textit{if } r \geq r_0, \end{cases}$$

where $r_0 = \exp(-3d/2)$ is such that $h_1(\cdot)$ is monotone increasing; and

$$h_2(r) = r^{d-2}\Big(\log\Big(\frac{1}{r}\Big)\Big)^{-c_{14}},$$

where $c_{14} > (3/\log 2 + 1)(d-2)$ is a constant.

Proof Let $\{B_i, 1 \leq i \leq m\}$ be an optimal cover for A, in the sense that

$$\nu_{h_1}(A, S_n) = \sum_{i=1}^m h_1\Big(\frac{s(B_i)}{2^n}\Big).$$

Write $r_i = s(B_i)$. If $r_i \geq n^{3d/2}$, then by Lemma 3.2, (d) of Lemma 2.1 and (2.2), and the definitions (2.4) and (2.5) of capacity, one can get that $\mathrm{Cap}_{g^\omega}(B_i \cap S_n) \leq c r_i^{d-2}$. On the other hand, if $r_i < n^{3d/2}$, then we enlarge the cube so that its side $r_i' = [n^{3d/2}]+1$, which has capacity bounded by $c(r_i')^{d-2} \leq c n^{3d(d-2)/2} \leq c(\log(2^n/r_i))^{3d(d-2)/2}$. It follows that for some constant $c_{13} > 0$, for all n sufficiently large,

$$\mathrm{Cap}_{g^\omega}(A) \leq \sum_{i=1}^m \mathrm{Cap}_{g^\omega}(B_i \cap S_n) \leq c_{13}\, 2^{n(d-2)} \sum_{i=1}^m h_1\Big(\frac{r_i}{2^n}\Big) = c_{13}\, 2^{n(d-2)} \nu_{h_1}(A, S_n).$$

Next we prove the lower bound in (3.10). By Lemma 2.5 there is a measure μ on A such that

$$\mu(A) \geq \nu_{h_2}(A, S_n) \quad \text{and} \quad \mu(V(x, 2^k)) \leq 2^d\, h_2\Big(\frac{2^k}{2^n}\Big) \quad \text{for all } 0 \leq k \leq n,\ x \in A. \tag{3.11}$$

For any $x \in A$, let $S_k(x) = V(x, 2^k) \backslash V(x, 2^{k-1})$. Let $c_0 = 3/\log 2 + 1$ so that for all n large enough, $2^{c_0 \log n} \geq c_{12} n^3$. Then by (d) and (e) of Lemma 2.1 and (2.2),

$$\begin{aligned}(g^\omega \mu)(x) &\leq \sum_{k=0}^{2n} \sum_{y \in A \cap S_k(x)} g^\omega(x, y)\mu(y) \\ &\leq \sum_{k=0}^{c_0 \log n} c_{10}\mu\big(S_k(x)\big) + \sum_{k=1+c_0 \log n}^{2n} \frac{c_9}{2^{k(d-2)}}\mu\big(S_k(x)\big).\end{aligned} \tag{3.12}$$

By using the second inequality in (3.11), and noting that $c_{14} > c_0(d-2)$ one can verify that

$$\sum_{k=0}^{c_0 \log n} c_{10}\mu\big(S_k(x)\big) \leq c\, 2^{-n(d-2)} \quad \text{and} \quad \sum_{k=1+c_0 \log n}^{2n} \frac{c_9}{2^{k(d-2)}}\mu\big(S_k(x)\big) \leq c\, 2^{-n(d-2)}.$$

This and (3.12) imply $(g^\omega \mu)(x) \leq c_{15} 2^{-n(d-2)}$ for all $x \in A$. Now we take the measure $\mu' = c_{15}^{-1} 2^{n(d-2)} \mu$. Then $(g^\omega \mu')(x) \leq 1$ for all $x \in A$. Therefore, by (2.6) and the first inequality in (3.11),

$$\mathrm{Cap}_{g^\omega}(A) \geq \mu'(A) = c_{15}^{-1} 2^{n(d-2)} \mu(A) \geq c_{15}^{-1} 2^{n(d-2)}\, \nu_{h_2}(A, S_n).$$

This proves the lower bound in (3.10). $\square$

3.2 Tail probability of the sojourn measure for the discrete time VSRW

In this subsection we focus on the discrete time VSRW $\{X_n,\, n = 0, 1, \ldots, \}$. For this process, for any $F \subseteq \mathbb{Z}^d$, the sojourn time of F is defined by

$$T(F) = \#\{n \geq 0 : X_n \in F\}. \tag{3.13}$$

The following lemma is an analogue of Lemma 7.6 in Barlow and Taylor [6] for random walks, which holds for any time-homogeneous Markov chains and can be proved similarly as in Lemma 3.1 in Pruitt and Taylor [25].

Lemma 3.5 *If $F \subseteq \mathbb{Z}^d$ satisfies*

$$M(F) := \sup_{y \in F} \mathrm{E}_y(T(F)) \in (0, \infty). \tag{3.14}$$

Then for every $\delta \in (0, 1)$ and any $\lambda \geq 0$, and for all $x \in \mathbb{Z}^d$,

$$\mathrm{P}_x\Big(T\big(F\big) \geq \lambda M(F)\Big) \leq e^{-\delta\lambda}.$$

Proof It suffices to prove that for all $x \in \mathbb{Z}^d$ and all integers $m \geq 1$,

$$\mathrm{E}_x\big(T(F)^m\big) \leq m!(M(F))^m. \tag{3.15}$$

This can be verified by using induction and the Markov property in a standard way (see, e.g., [25]). We omit the details. $\square$

Next we estimate $M(F)$. Denote by $V_k(y) = V(y, 2^k)$ the cube in $\mathbb{Z}^d$ centered at y with side 2^k. Let c_{17} be a large constant so that

$$2^k \geq 2c_{11}n^3 \quad \text{and} \quad 2^{2k} \geq n^{3d} \quad \text{for all } k \geq c_{17}\log n > 0. \tag{3.16}$$

Lemma 3.6 *Assume that $d \geq 3$ and $\mathbb{P}(\mu_e \geq 1) = 1$. Then there exists a constant c_{16} such that $\mathbb{P}$-a.s. for all n large enough, and $c_{17}\log n \leq k \leq n$ the inequality*

$$\mathrm{E}^\omega_x\big(T(V_k(y))\big) \leq c_{16}\, 2^{2k} \tag{3.17}$$

holds uniformly for all $x, y \in V(0, 2^{n+1})$.

Proof As in (3.4), we have that for any $F \subseteq \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$,

$$\mathrm{E}^\omega_x\big(T(F)\big) = \sum_{z\in F} \widehat{g}^\omega(x,z).$$

Moreover, by (2.2) $\max_{x\in V_{n+1}} U_x \leq c_{11}\, n^3$. It follows from Lemma 2.2 that

$$\begin{aligned}\mathrm{E}^\omega_x\big(T(V_k(y))\big) &\leq \sum_{z\in V(x,c_{11}n^3)} \widehat{g}^\omega(x,z) + \sum_{z\in V(y,2^k)\setminus V(x,c_{11}n^3)} \frac{c_9}{|x-z|^{d-2}}\\ &\leq C\left(n^{3d} + 2^{2k}\right) \leq c_{16}\, 2^{2k}.\end{aligned}$$

This proves (3.17). $\square$

It follows from Lemma 3.6 that $\mathbb{P}$-a.s. for all n large enough, $c_{17}\log n \leq k \leq n$ and all $x \in V_n$,

$$M(Q_k(x)) \leq c_{16}\, 2^{2k}, \tag{3.18}$$

where, recall that, for any $x \in \mathbb{Z}^d$ and any $0 \leq k \in \mathbb{Z}$, $Q_k(x)$ is the unique cube in $\mathcal{C}^k_d$ that contains x.

3.3 A maximal inequality

The following lemma estimates the tail probability of the maximal displacement of VSRW X.

Lemma 3.7 *Assume that $d \geq 3$ and $\mathbb{P}(\mu_e \geq 1) = 1$. Then there exist constants c_{18}, c_{19} and c_{20} such that $\mathbb{P}$-a.s. for all n large enough ($n \geq n_0$) the inequality*

$$\mathrm{P}^\omega_x\left(\sup_{0\leq t\leq T} |X_t - x| > \lambda\sqrt{T}\right) \leq c_{18}\exp\big(-c_{19}\lambda^2\big) \tag{3.19}$$

holds for all $x \in V(0, 2^{n-1})$, $(c_{11}n^3)^2 \leq T \leq 2^n$ and $c_{20} \leq \lambda < \sqrt{T}/2$.

Proof For any n and $T, a > 0$, let

$$\alpha(T,a) = \sup_{||x||_\infty \le 2^n,\, 0\le t\le T} \mathrm{P}_x^\omega\big(|X_t - X_0| > a\big). \tag{3.20}$$

For any $x \in V(0, 2^{n-1})$ and $2 \le M < 2^{n-1}$, we consider the stopping time

$$\tau = \inf\{t > 0 : ||X_t - x||_\infty > M\}.$$

For VSRW X started at x, we have $||X_\tau - x||_\infty \le M+1$ and hence $||X_\tau||_\infty \le 2^{n-1}+M+1 \le 2^n$. The triangle inequality and the strong Markov property imply that

$$\mathrm{P}_x^\omega\left(|X_T - x| > \frac{M}{2}\right) \ge \mathrm{E}_x^\omega\left[\mathrm{P}_{X_\tau}^\omega\left(|X_{T-\tau} - X_0| \le \frac{M}{2}\right) \mathbb{1}_{\{\tau \le T\}}\right]. \tag{3.21}$$

By the definition (3.20) and that $||X_\tau||_\infty \le 2^n$, for any pair (τ, X_τ),

$$\begin{aligned}\mathrm{P}_{X_\tau}^\omega\left(|X_{T-\tau} - X_0| \le \frac{M}{2}\right) &= 1 - \mathrm{P}_{X_\tau}^\omega\left(|X_{T-\tau} - X_0| > \frac{M}{2}\right)\\ &\ge 1 - \alpha(T, M/2).\end{aligned} \tag{3.22}$$

Hence we derive the following quenched Ottaviani-type inequality: For all $x \in V(0, 2^{n-1})$, $2 \le M < 2^{n-1}$ and $T > 0$ such that $\alpha(T, M/2) < 1$,

$$\mathrm{P}_x^\omega\left(\sup_{0\le t\le T} |X_t - x| > M\right) \le \frac{\mathrm{P}_x^\omega\big(|X_T - x| > M/2\big)}{1 - \alpha(T, M/2)}. \tag{3.23}$$

This is reminiscent to Lemma 2 in Gikhman and Skorohod [16, p.420].

Next, recall that $\mathbb{P}$-a.s. for all $n \ge n_0$, we have $\max_{||x||_\infty \le 2^n} U_x \le c_{11} n^3$, see (2.2). If $a > c_{11} n^3$, then we can apply (b) of Lemma 2.1 to deduce that for all $x \in V(0, 2^n)$ and all $0 \le t \le T$ with $t \le a$,

$$\begin{aligned}\mathrm{P}_x^\omega\big(|X_t - x| > a\big) &= \sum_{y\in\mathbb{Z}^d : |y-x|>a} p_t^\omega(x,y)\\ &\le c_4 \sum_{y\in\mathbb{Z}^d : |y-x|>a} \exp\Big(-c_5|y-x|\Big)\\ &\le c_{21}\, e^{-c_{22} a}.\end{aligned} \tag{3.24}$$

If $T > a\,(> c_{11} n^3)$ and $a < t \le T$, then it can be verified that for all $x \in V(0, 2^{n-1})$,

$$\begin{aligned}\mathrm{P}_x^\omega\big(|X_t - x| > a\big) &= \sum_{a<|y-x|\le t} p_t^\omega(x,y) + \sum_{|y-x|>t} p_t^\omega(x,y)\\ &\le c_4 \sum_{a<|y-x|\le t} t^{-d/2} \exp\Big(-c_5|y-x|^2/t\Big) + c_{21} e^{-c_{22} t}\\ &\le c_{23} e^{-c_{24}(a/\sqrt{t})^2} + c_{21} e^{-c_{22} t}.\end{aligned} \tag{3.25}$$

Now we apply (3.23) with $M = \lambda\sqrt{T}$, where $(c_{11}n^3)^2 \le T < 2^n$ and $c_{20} \le \lambda \le \sqrt{T}/2$. It follows from (3.24) and (3.25) that we can choose n and the constant c_{20} large enough such that

$$\alpha(T,\, M/2) \le \frac{1}{2}. \tag{3.26}$$

By (3.25), we have that for all $\lambda \le \sqrt{T}/2$,

$$\mathrm{P}_x^\omega\Big(|X_T - x| > \lambda\sqrt{T}\Big) \le c' e^{-c''\lambda^2}. \tag{3.27}$$

Plugging (3.26) and (3.27) into (3.23) yields (3.19). □

It is known that VSRW spends a time of order n^2 in the cube $V(0,n)$ (Barlow and Černý [3, p.655]). By applying Lemma 3.7 and the Borel-Cantelli lemma we obtain

Corollary 3.8 *For $\mathbb{P}$-a.e. ω, P_0^ω-a.s.,*

$$\limsup_{T\to\infty} \frac{\max_{0\le t\le T}|X_t|}{\sqrt{T\log\log T}} \le \frac{1}{\sqrt{c_{19}}}.$$

Consequently, the time that VSRW X spends in the cube $V(0,n)$ is at least $cn^2/\sqrt{\log\log n}$.

3.4 A zero-one law

Lemma 3.9 *For any (infinite) set $A \subset \mathbb{Z}^d$, for $\mathbb{P}$-almost every $\omega \in \Omega$,*

$$\mathrm{P}_0^\omega\big(X_t \in A \ \ \textit{for arbitrarily large } t > 0\big) \in \{0,1\}.$$

Proof T his is a consequence of an elliptic Harnack inequality that the VSRW X satisfies. More explicitly, define

$$h(x) = \mathrm{P}_x^\omega\big(X_t \in A \ \ \text{for arbitrarily large } t > 0\big).$$

Then h is a harmonic function (with respect to the generator $\mathcal{L}_V$ in (1.1)). Let $a = \inf_x h(x) \ge 0$, and let $g = h - a$, so $g \ge 0$ with $\inf_x g(x) = 0$. If g is not identically zero, then there exists x_0 such that $g(x_0) > 0$. Now for any $R \ge U_{x_0}$, by Corollary 4.8 in Barlow and Deuschel [4],

$$g(x_0) \le \sup_{x\in B(x_0,R/2)} g(x) \le C \inf_{x\in B(x_0,R/2)} g(x).$$

This holds for all $R \ge U_{x_0}$, so one gets

$$g(y) \ge g(x_0)/C \text{ for all } y,$$

a contradiction to that $\inf_x g(x) = 0$. So h must be a constant function. On the other hand, the martingale convergence theorem tells us that P_0^ω almost surely,

$$h(X_t) \to \mathbb{1}_{\{X_t\in A \ \text{ for arbitrarily large } t>0\}} \quad \text{as } t\to\infty.$$

So h is either constantly 0 or constantly 1. □

3.5 Proofs of Theorems 1.1 and 1.2

With the results established above, we are ready to prove Theorems 1.1 and 1.2. Even though the arguments are similar to the proofs of Theorem 7.8 and Theorem 8.3 of Barlow and Taylor [6], several modifications are needed.

Proof *of Theorem 1.1* Firstly we prove that for $\mathbb{P}$-a.e. ω, the packing dimension of the range $\dim_{\mathrm{P}} \mathrm{R} \leq 2$ P_0^ω-a.s. This is done by using a first moment argument.

Let M_k be the total number of semi-dyadic cubes in $\mathcal{C}_s^k$ of order k which are contained in S_n and are hit by $\{X_t, t \geq 0\}$. Since there are at most $c2^{(n-k)d}$ semi-dyadic cubes of order k contained in S_n and, by Lemma 3.2, for all k such that $n^{3d/2} \leq 2^k \leq 2^{n-1}$, or equivalently, $c \log n \leq k \leq n-1$ for some $c > 0$, each of them can be hit by X with probability at most $c2^{(d-2)(k-n)}$. Hence

$$\mathrm{E}_0^\omega\big(M_k\big) \leq c\, 2^{2(n-k)}, \quad \text{for all } c \log n \leq k \leq n-1. \tag{3.28}$$

Now we take $\delta > 0$ small, and $\beta = 2 + \delta$. It follows from (2.15) and (3.28) that for $\varepsilon \in (0,1)$,

$$\begin{aligned} \mathrm{E}_0^\omega\Big(\widetilde{\tau}_\beta\big(\mathrm{R}, S_n, \varepsilon\big)\Big) &\leq \sum_{k=1}^{c\log n} c2^{(n-k)d} \cdot 2^{(d-2)(c\log n - n)} \cdot \Big(\frac{2^k}{2^n}\Big)^\beta + c \sum_{k=c\log n}^{n(1-\varepsilon)} 2^{2(n-k)} \Big(\frac{2^k}{2^n}\Big)^\beta \\ &\leq c\, 2^{(d-2)c\log n} \cdot 2^{-n\varepsilon\delta}. \end{aligned} \tag{3.29}$$

Hence $\widetilde{p}_\beta\big(\mathrm{R}, \varepsilon\big) < \infty$ for all $\varepsilon \in (0,1)$ P_0^ω-a.s. This and the arbitrariness of $\delta > 0$ imply $\dim_{\mathrm{P}} \mathrm{R} \leq 2$ P_0^ω-a.s.

Secondly we prove $\dim_{\mathrm{H}} \mathrm{R} \geq 2$ P_0^ω-a.s. Let $\widehat{\mathrm{R}}$ be the range of the discrete time VSRW $\widehat{X}$:

$$\begin{aligned} \widehat{\mathrm{R}} &:= \big\{x \in \mathbb{Z}^d : \widehat{X}_n = x \quad \text{for some } n \geq 0\big\} \\ &= \{x \in \mathbb{Z}^d : X_n = x \quad \text{for some } n \geq 0\}. \end{aligned}$$

Since $\widehat{\mathrm{R}}$ is a subset of R, it suffices to show that $\dim_{\mathrm{H}} \widehat{\mathrm{R}} \geq 2$ P_0^ω-a.s. Let μ be the measure on $\widehat{\mathrm{R}}$ which assigns mass 1 to each point of $\widehat{\mathrm{R}}$. We claim that there is a constant c_{27} such that P_0^ω-a.s. for all n large enough

$$\mu\big(Q_k(x)\big) \leq c_{27}\, n\, 2^{2k} \qquad \text{for every } x \in S_n \text{ and } 0 \leq k \leq n. \tag{3.30}$$

Note that the above inequality holds automatically for all $x \in S_n$ and $0 \leq k \leq 1/(d-2) \cdot \log n$. A simple covering argument shows that it also holds for all $x \in S_n$ and $0 \leq k \leq c_{17} \log n$, where c_{17} is the constant in (3.16). Hence, in order to prove (3.30), it is sufficient to consider the event

$$E_n = \Big\{\mu\big(Q_k(x)\big) > \gamma\, n\, 2^{2k} \text{ for some } x \in S_n \text{ and } c_{17} \log n \leq k \leq n\Big\} \tag{3.31}$$

and show that $\sum_{n=1}^\infty \mathrm{P}_0^\omega(E_n) < \infty$. In the above, $\gamma > 0$ is a generic constant whose value will be chosen later.

Since $\mu\big(Q_k(x)\big) > 0$ implies $Q_k(x)$ is hit by $\widehat{X}$, it follows from Lemma 3.2 that for all $c_{17} \log n \leq k \leq n-3$ (note that by (3.16), $2^{c_{17}\log n} \geq n^{3d/2}$, hence Lemma 3.2 applies), for every $x \in S_n$,

$$\mathrm{P}_0^\omega\big(\widehat{X}_n \in Q_k(x) \text{ for some } n\big) \leq \mathrm{P}_0^\omega\big(X_t \in Q_k(x) \text{ for some } t\big) \leq c2^{(k-n)(d-2)}.$$

By enlarging c if necessary we can assume that the above inequality also holds for $k = n-2$, $n-1$ and n. Moreover, restarting at the hitting point (say, $\widehat{X}_\tau$, which necessarily lies in V_{n+1}) and applying Lemma 3.5 and (3.18), we have that for all n large enough,

$$\mathrm{P}^\omega_{\widehat{X}_\tau}\Big(\mu\big(Q_k(x)\big) > \gamma\, n\, 2^{2k}\Big) \le \mathrm{P}^\omega_{\widehat{X}_\tau}\Big(T\big(Q_k(x)\big) > \gamma\, n\, 2^{2k}\Big) \le e^{-\delta\gamma\, n}, \qquad \mathrm{P}^\omega_0\text{-a.s.},$$

where $\delta \in (0,1)$ is a constant. Note that when applying (3.18) we have again used the fact that $k \ge c_{17}\log n$.

It follows from the above and the strong Markov property that

$$\begin{aligned} \mathrm{P}^\omega_0\big(E_n\big) &\le \sum_{c_{17}\log n \le k \le n} \mathrm{P}^\omega_0\Big\{\mu\big(Q_k(x)\big) > \gamma n\, 2^{2k} \text{ for some } x \in S_n\Big\} \\ &\le \sum_{c_{17}\log n \le k \le n} c\, 2^{(n-k)d} \cdot 2^{(k-n)(d-2)} \cdot e^{-\gamma\,\delta n} \\ &\le c\, e^{-(\gamma\,\delta - 2)n}, \end{aligned} \tag{3.32}$$

for all n large enough. We now take $\gamma > 2/\delta$ so that $\sum_{n=1}^\infty \mathrm{P}^\omega_0(E_n) < \infty$. This and the Borel-Cantelli lemma prove (3.30).

Hence, by (3.30) and Lemma 2.4, we have

$$\nu_2\big(\widehat{\mathrm{R}}, S_n\big) \ge 2^{-d}\, c_{27}^{-1}\, n^{-1} 2^{-2n}\, \mu(S_n) \tag{3.33}$$

for all n large enough. By Lemma 3.3 we have

$$\mathrm{E}^\omega_0\big(\mu(S_n)\big) = \sum_{y \in S_n} \mathrm{P}^\omega_0\big(\widehat{X}_i = y \text{ for some } i \ge 0\big) \ge c_{28}\, 2^{2n} \tag{3.34}$$

for all n large. This, (3.33) and (2.10) imply $\mathrm{E}^\omega_0\big(m_2(\widehat{\mathrm{R}})\big) = \infty$.

Next we prove that $m_2\big(\widehat{\mathrm{R}}\big) = \infty$ P^ω_0-a.s. By (3.15) and (3.18), we have

$$\mathrm{E}^\omega_0\big(\mu(S_n)^2\big) \le \mathrm{E}^\omega_0\big(T(S_n)^2\big) \le 2\Big(M(S_n)\Big)^2 \le c_{29} 2^{4n}. \tag{3.35}$$

Thus, by the Paley-Zygmund inequality ([17, p.8]), we obtain

$$\begin{aligned} \mathrm{P}^\omega_0\Big(\mu(S_n) \ge \frac{1}{2} c_{28} 2^{2n}\Big) &\ge \mathrm{P}^\omega_0\Big(\mu(S_n) \ge \frac{1}{2}\mathrm{E}^\omega_0\big(\mu(S_n)\big)\Big) \\ &\ge \frac{1}{4}\frac{\Big(\mathrm{E}^\omega_0\big(\mu(S_n)\big)\Big)^2}{\mathrm{E}^\omega_0\big(\mu(S_n)^2\big)} \\ &\ge \frac{(c_{28})^2}{4\, c_{29}} := p. \end{aligned}$$

Moreover, we can replace P^ω_0 by P^ω_x and use Lemma 3.6 and the same argument as above to show that the inequality

$$\mathrm{P}^\omega_x\Big(\mu(S_n) \ge \frac{1}{2} c_{28}\, 2^{2n}\Big) \ge p \tag{3.36}$$

holds uniformly for all n large and for all $x \in V(0, 2^{n-2})$.

We let $n_k = \lfloor \lambda k \log k \rfloor$, where $\lambda > 0$ denotes a constant whose value will be chosen later, and define a sequence of stopping times by

$$\tau_k = \inf\Big\{ n > 0 : \ \widehat{X}_n \notin V\big(0, 2^{n_k}\big)\Big\}, \quad (k \geq 1). \tag{3.37}$$

Note that $|\widehat{X}_{\tau_{k+1}}| \geq 2^{n_{k+1}}/2$, hence by using the strong Markov property and Lemma 3.2 we obtain that P_0^ω almost surely,

$$\mathrm{P}^\omega_{\widehat{X}_{\tau_{k-1}}}\Big(\widehat{X}_n \in S_{n_k} \text{ for some } \ n \geq \tau_{k+1}\Big) \leq c\bigg(\frac{2^{n_k}}{2^{n_{k+1}}}\bigg)^{d-2} \leq \frac{1}{k^2}, \tag{3.38}$$

when the constant λ is chosen large enough.

Next we consider

$$\begin{aligned} \mathrm{P}_0^\omega\Big(|\widehat{X}_{\tau_{k-1}}| > 2^{n_k-3}\Big) \leq \mathrm{P}_0^\omega\Big(&|\widehat{X}_{\tau_{k-1}}| > 2^{n_k-3}, \ \tau_{k-1} \leq 2^{2n_{k-1}} n_{k-1}\Big) \\ &+ \mathrm{P}_0^\omega\Big(\tau_{k-1} > 2^{2n_{k-1}} n_{k-1}\Big). \end{aligned} \tag{3.39}$$

Lemma 3.7 implies that

$$\begin{aligned} \mathrm{P}_0^\omega\Big(|\widehat{X}_{\tau_{k-1}}| > 2^{n_k-3}, \ \tau_{k-1} \leq 2^{2n_{k-1}} n_{k-1}\Big) &\leq \mathrm{P}_0^\omega\bigg(\sup_{0 \leq t \leq 2^{2n_{k-1}} n_{k-1}} |X_t| > 2^{n_k-3}\bigg) \\ &\leq c_{18} \exp\bigg(- c_{19} \, \frac{(k-1)^{2\log(2)\lambda}}{64\lambda(k-1)\log(k-1)}\bigg) \\ &\leq c_{18} \exp(-c_{29} k) \end{aligned} \tag{3.40}$$

for all k large enough when λ is chosen large enough. On the other hand, by Lemmas 3.5 and 3.6 we have

$$\mathrm{P}_0^\omega\Big(\tau_{k-1} > 2^{2n_{k-1}} n_{k-1}\Big) \leq \mathrm{P}_0^\omega\Big(T\big(V(0, 2^{n_{k-1}})\big) \geq 2^{2n_{k-1}} n_{k-1}\Big) \leq c_{18} \exp(-c_{29} k) \tag{3.41}$$

for all k large enough, again when λ is chosen large enough. Combining (3.39), (3.40) and (3.41) yields

$$\mathrm{P}_0^\omega\Big(|\widehat{X}_{\tau_{k-1}}| > 2^{n_k-3}\Big) \leq 2c_{18} \exp(-c_{29} k) \tag{3.42}$$

for all k large enough.

By (3.36) we have that P_0^ω-a.s. on the event $\big\{|\widehat{X}_{\tau_{k-1}}| \leq 2^{n_k-3}\big\}$,

$$\mathrm{P}^\omega_{\widehat{X}_{\tau_{k-1}}}\Big(\mu(S_{n_k}) \geq \frac{1}{2} c_{28}\, 2^{2n_k}\Big) \geq p. \tag{3.43}$$

Define

$$\widehat{\mathrm{R}}_k = \big\{ x \in \mathbb{Z}^d : \widehat{X}_n = x \ \text{ for some } \ \tau_{k-1} \leq n < \tau_{k+1}\big\}$$

to be the range of the discrete time VSRW $\widehat{X}$ between the times τ_{k-1} and τ_{k+1}. Noting that $\widehat{X}_\ell$ does not belong to S_{n_k} when $\ell < \tau_{k-1}$, and using (3.38), the strong Markov property and (3.43), one obtains that for k large, on the event $\big\{|\widehat{X}_{\tau_{k-1}}| \leq 2^{n_k-3}\big\}$,

$$\mathrm{P}^\omega_{\widehat{X}_{\tau_{k-1}}}\Big(\mu(\widehat{\mathrm{R}}_k \cap S_{n_k}) \geq \frac{1}{2} c_{28}\, 2^{2n_k}\Big) \geq p - \frac{1}{k^2} \geq \frac{p}{2}. \tag{3.44}$$

Using (3.44) and (3.42) and applying Lemma 2.6 we conclude that P_0^ω-a.s. the inequality

$$\mu(\widehat{\mathrm{R}}_{2k} \cap S_{n_{2k}}) \geq \frac{1}{2} c_{28}\, 2^{2n_k}$$

holds for a sequence K of integers k of lower density at least ε for some constant $\varepsilon > 0$. This and (3.33) imply

$$\nu_2\big(\widehat{\mathrm{R}}_{2k}, S_{n_{2k}}\big) \geq c_{30} n_{2k}^{-1}, \quad \text{for all } k \in K.$$

Therefore,

$$m_2(\widehat{\mathrm{R}}) \geq c_{30} \sum_{k \in K} \frac{1}{n_{2k}} \geq c_{31} \sum_{k \in K} \frac{1}{2k \log(2k)} = \infty, \qquad \mathrm{P}_0^\omega\text{-a.s.}$$

This proves that $\dim_{\mathrm{H}} \widehat{\mathrm{R}} \geq 2$ P_0^ω-a.s. and the theorem. □

Proof *of Theorem 1.2* It follows from (2.4), (2.5) and (d) of Lemma 2.1 that for all n large enough,

$$\mathrm{P}_0^\omega\Big(X_t \in A \cap S_n \ \text{ for some } \ t > 0\Big) \asymp \frac{1}{2^{n(d-2)}} \mathrm{Cap}_{g^\omega}(A \cap S_n). \tag{3.45}$$

Hence the proof of Theorem in Lamperti [22] gives that

$$\mathrm{P}_0^\omega\Big(X_t \in A \ \text{ for arbitrarily large}\, t > 0\Big) = 0$$

if and only if $\sum_{n=1}^\infty 2^{-n(d-2)} \mathrm{Cap}_{g^\omega}(A \cap S_n) < \infty$. The set S_n here and the S_n in [22] have different meanings, nevertheless it is straightforward to modify the arguments in [22] to our setting. The assumption (7) therein should be modified to: there exist $a, b < \infty$ (depending on the environment) such that for any $x \in S_n$, $y \in S_{n+m}$ where $n, m \geq b$,

$$g^\omega(x, y) \leq a 2^{-(n+m)(d-2)}, \quad g^\omega(y, x) \leq a 2^{-n(d-2)}.$$

This holds thanks to (d) of Lemma 2.1 and (2.2).

Combining this with Lemma 3.4, we deduce

- If $m_{h_1}(A) < \infty$, then $\mathrm{P}_0^\omega\big(X_t \in A \ \text{ for arbitrarily large}\, t > 0\big) = 0$.
- If $m_{h_2}(A) = \infty$, then $\mathrm{P}_0^\omega\big(X_t \in A \ \text{ for arbitrarily large}\, t > 0\big) > 0$.

These and Lemma 3.9 imply the conclusions of Theorem 1.2. □

4 Proof of Theorem 1.4

The proofs of Theorems 1.1 and 1.2 only make use of Lemmas 2.1 and 3.9. Bounds on the Green's function are used for estimating the hitting probabilities in Sections 3.1 and 3.2; and the upper bound on the transition density $p_t^\omega(x, y)$ is used to derive the maximal inequality in Section 3.3.

Proof *of Theorem 1.4* If $a = 0$, $\tilde{X}$ is a time change of the simple random walk on $\mathbb{Z}^d$. Hence Theorem 1.4 follows from Theorems 7.8 and 8.3 in Barlow and Taylor [6].

Assume $a \in (0, 1]$. It follows from Lemma 9.1 of Barlow and Černý [3] and Theorem 6.1 in Barlow and Deuschel [4] that, under the assumption that $\tilde{\mathbb{P}}(\kappa_x \geq 1) = 1$, the transition density and the Green's function of the VSRW $\tilde{X}$ satisfy Lemma 2.1. Moreover, by Lemma 9.1 and Proposition 3.2 in Barlow and Černý [3], $\tilde{X}$ also enjoys an elliptic Harnack inequality and hence a zero-one law as in Lemma 3.9 holds as well. Therefore, the proof of Theorem 1.4 is the same as those of Theorems 1.1 and 1.2. $\square$

A Appendix: Proof of Lemma 2.3

We first list some known facts about discrete time Markov chains with discrete space E. Suppose $\{X_i,\, i \geq 0\}$ is such a process, for any finite set A in the state space, let

$$T_A = \inf\{i \geq 0 : X_i \in A\}, \quad \text{and} \quad S_A = \inf\{i > 0 : X_i \in A\} \tag{A.1}$$

be the first hitting time and the first return time of A respectively. Then by the last-exit decomposition, see, e.g., Proposition 3.5 in Revuz [26, p.57]

$$\mathrm{P}_x(T_A < \infty) = \sum_{y \in A} g(x, y)\mathrm{P}_y(S_A = \infty), \qquad \text{for all } x,$$

where $g(x, y) = \sum_{i=0}^{\infty} p_i(x, y)$ is the Green's function. Moreover, if we define

$$\mathrm{Cap}(A) = \sum_{y \in A} P_y(S_A = \infty)$$

to be the capacity of A, then it satisfies that Revuz [26, Exercise 4.13 on p.64]

$$\mathrm{Cap}(A) = \max\Big\{\sigma(A) : \sigma \text{ is a measure on } A \text{ such that } \max_{x \in A}(g\sigma)(x) \leq 1\Big\}. \tag{A.2}$$

We now prove Lemma 2.3. For any n, define discrete time Markov chain $\{X_i^{(n)} := X_{i/2^n},\, i = 0, 1, \ldots\}$. It has transition density $p_i^{(n)}(x, y) = p_{i/2^n}(x, y)$, and Green's function $g^{(n)}(x, y) = \sum_{i=0}^{\infty} p_{i/2^n}(x, y)$. Now for any finite set A in the state space E, define the hitting time and return time $T_A^{(n)}$ and $S_A^{(n)}$ similarly as in (A.1) for the process $X^{(n)}$. We then have that

$$\mathrm{P}_x\Big(T_A^{(n)} < \infty\Big) = \sum_{y \in A} g^{(n)}(x, y)\mathrm{P}_y\Big(S_A^{(n)} = \infty\Big), \quad \text{for all } x. \tag{A.3}$$

Moreover, $\mathrm{Cap}^{(n)}(A) = \sum_{y \in A} \mathrm{P}_y(S_A^{(n)} = \infty)$ satisfies (A.2) with g replaced by $g^{(n)}$.

We now let n go to ∞. By the right continuity of the process $\{X_t, t \geq 0\}$,

$$\mathrm{P}_x(T_A^{(n)} < \infty) \uparrow \mathrm{P}_x(T_A < \infty), \text{ where } T_A = \inf\{t \geq 0 : X_t \in A\}. \tag{A.4}$$

Moreover, by condition (ii) and Lemma 3.6.1 in Norris [24], for any $x, y \in E$ and any $0 \leq s < t < \infty$,

$$|p_s(x, y) - p_t(x, y)| \leq 1 - e^{-q_x(t-s)} = O(t - s).$$

Combining this with condition (i) one can verify that

$$\frac{1}{2^n} g^{(n)}(x,y) \to g(x,y) = \int_0^\infty p_t(x,y)\,dt, \quad \text{for all } x, y \in E. \tag{A.5}$$

Furthermore, by condition (ii) again, for any $y \in A$ and $n \geq 1$,

$$2^n \mathrm{P}_y(S_A^{(n)} = \infty) \leq 2^n \mathrm{P}_y(X_{1/2^n} \neq y) \leq q_y < \infty,$$

hence $\{2^n \mathrm{P}_y(S_A^{(n)} = \infty) : n \geq 1\}$ must admit a subsequence converging to some limit, say $b(y)$. By (A.3), (A.4) and (A.5) the $b(y)$'s must satisfy

$$\mathrm{P}_x(T_A < \infty) = \sum_{y \in A} g(x,y) b(y), \quad \text{for all } x \in E.$$

Thus we have explicitly built a function $b(y)$ which solves (2.4). Moreover, by the uniqueness of Riesz decomposition (see, for example, Syski [28, Theorem 1, p.165]), the solution to the above equation is unique, and hence we conclude that the whole sequence $\{2^n \mathrm{P}_y(S_A^{(n)} = \infty)\}$ must converge to $b(y)$. We then have that

$$\mathrm{Cap}(A) = \lim_{n \to \infty} 2^n \mathrm{Cap}^{(n)}(A).$$

That it satisfies (2.6) follows from the above convergence and that $\mathrm{Cap}^{(n)}(A)$ satisfies (A.2) .

Finally the lemma applies to the VSRW, because condition (i) holds thanks to (a) of Lemma 2.1, and (ii) holds by the definition of VSRW. □

Acknowledgments

We thank Martin Barlow for the proof of Lemma 3.9. We also thank the editor and an anonymous referee for their helpful and thoughtful comments and suggestions.

References

[1] S. Andres, M. T. Barlow, J.-D. Deuschel and B. Hambly, Invariance principle for the random conductance model. *Preprint*, 2010.

[2] M. T. Barlow, Random walks on supercritical percolation clusters. *Ann. Probab.* **32** (2004), 3024–3084.

[3] M. T. Barlow and J. Černý, Convergence to fractional kinetics for random walks associated with unbounded conductances. *Probab. Th. Rel. Fields* **149** (2011), 639–673.

[4] M. T. Barlow and J.-D. Deuschel, Invariance principle for the random conductance model with unbounded conductance. *Ann. Probab.* **38** (2010), 234–276.

[5] M. T. Barlow and S. J. Taylor, Fractal dimension of sets in discrete spaces. *J. Phys. A* **22** (1989), 2621–2626.

[6] M. T. Barlow and S. J. Taylor, Defining fractal subsets of $\mathbb{Z}^d$. *Proc. London Math. Soc.* **64** (1992), 125–152.

[7] M. T. Barlow and X. Zheng, The random conductance model with Cauchy tails. *Ann. Appl. Probab.* **20** (2010), 869–889.

[8] G. Ben Arous and J. Černý, Scaling limit for trap models on $\mathbb{Z}^d$. *Ann. Probab.* **35** (2007), 2356–2384.

[9] N. Berger, M. Biskup, C. E. Hoffman and G. Kozma, Anomalous heat-kernel decay for random walk among bounded random conductances. *Ann. Inst. H. Poincaré Probab. Statist.* **44** (2008), 374–392.

[10] M. Biskup and T. M. Prescott, Functional CLT for random walk among bounded random conductances. *Electron. J. Probab.* **12** (2007), 1323–1348.

[11] J. Černý, On two-dimensional random walk among heavy-tailed conductances. *Electron. J. Probab.* **16** (2011), 293–313.

[12] K. L. Chung, Probabilistic approach in potential theroy to the equilibrium. *Ann. Inst. Fourier Grenoble* **23** (1973), 313–322.

[13] K. L. Chung, Remarks on equilibrium potential and erengy. *Ann. Inst. Fourier Grenoble* **25** (1975), 131–138.

[14] T. Delmotte, Parabolic Harnack inequality and estimates of Markov chains on graphs. *Rev. Mat. Iberoamericana* **15** (1999), 181–232.

[15] K. J. Falconer, *Fractal Geometry – Mathematical Foundations and Applications.* Wiley & Sons, New York, 1990.

[16] I. I. Gikhman and A. V. Skorohod, *The Theory of Stochastic Processes* **1**. Springer, Berlin, 1974.

[17] J.-P. Kahane, *Some Random Series of Functions.* 2nd edition, Cambridge University Press, Cambridge, 1985.

[18] O. Kallenberg, *Foundations of Modern Probability,* 2nd ed. Springer, New York, 2004.

[19] D. Khoshnevisan, A discrete fractal in $\mathbb{Z}^1_+$. *Proc. Amer. Math. Soc.* **136** (1994), 577–584.

[20] D. Khoshnevisan, Escape rates for Lévy processes. *Studia Sci. Math. Hungarica* **33** (1997), 177–183.

[21] G. F. Lawler and V. Limic, *Random Walk: A Modern Introduction.* Cambridge University Press, Cambridge, 2010.

[22] J. Lamperti, Wiener's test and Markov chains. *J. Math. Anal. Appl.* **6** (1963), 58–66.

[23] P. Mathieu, Quenched invariance principles for random walks with random conductances. *J. Stat. Phys.* **130** (2008), 1025–1046.

[24] J. R. Norris, *Markov Chains*, Cambridge University Press, Cambridge, 1998.

[25] W. E. Pruitt and S. J. Taylor, Sample path properties of processes with stable components. *Z. Wahrsch. Verw. Gebiete* **12** (1969), 267–289.

[26] D. Revuz, *Markov Chains*, Elsevier Science Publishers B. V., Amsterdam, 1984.

[27] F. Solomon, Random walks in a random environment, *Ann. Probab.* **3** (1975), 1–31.

[28] R. Syski, *Passage Times for Markov Chains.* IOS Press, Amsterdam, 1992.

[29] S. J. Taylor, The measure theory of random fractals. *Math. Proc. Camb. Phil. Soc.* **100** (1986), 383–406.

[30] Y. Xiao, Asymptotic results for self-similar Markov processes. In: *Asymptotic Methods in Probability and Statistics (ICAMPS'97)* (B. Szyszkowicz, Editor), pp. 323–340, Elsevier Science, 1998.

[31] Y. Xiao, Random fractals and Markov processes. In: *Fractal Geometry and Applications: A Jubilee of Benoit Mandelbrot*, (Michel L. Lapidus and Machiel van Frankenhuijsen, editors), pp. 261–338, American Mathematical Society, 2004.

[32] Y. Xiao, Sample path properties of anisotropic Gaussian random fields. In: *A Minicourse on Stochastic Partial Differential Equations*, Khoshnevisan, D., Rassoul-Agha, F. (Eds.), Lecture Notes in Math. **1962**, pp. 145–212, Springer, New York, 2009.

Yimin Xiao: Department of Statistics and Probability, A-413 Wells Hall, Michigan State University, East Lansing, MI 48824, U.S.A.
E-mail: `xiao@stt.msu.edu`

Xinghua Zheng: Department of ISOM, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong
E-mail: `xhzheng@ust.hk`